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Exact and Explicit Solitary Wave Solutions to Some Nonlinear Equations

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Exact and explicit solitary wave solutions are obtained for some physically interesting nonlinear evolutions and wave equations in physics and other fields by using a special transformation. These equations include the KdV–Burgers equation, the MKdV–Burgers equation, the combined KdV–MKdV equation, the Newell–Whitehead equation, the dissipative Φ^4-model equation, the generalized Fisher equation, and the elastic-medium wave equation.

Nonlinear evolutions and wave equations are special classes of partial differential equations which have been studied intensively in recent decades. Searching for exact and explicit solitary wave solutions of these equations has long been a major concern for both mathematicians and physicists. Although some significant and successful methods have been developed (Drazin and Johnson, 1989; Sachdev, 1987; Huang *et al.*, 1989; Lou, 1991; Newell and Moloney, 1992; Lu *et al.*, 1993; Ablowitz and Zeppetella, 1979; Murray, 1989; Hereman *et al.*, 1986; Hereman and Takaoka, 1990; Coffey, 1990, 1992; Wang, 1988; Wang *et al.*, 1990; Yang *et al.*, 1994; Yang, 1994; Ma, 1993), there unfortunately exists no general method for obtaining exact and explicit solutions of nonlinear evolutions and wave equations. In this paper we present a new method for obtaining exact and explicit solitary wave solutions of some nonlinear evolutions and wave equations. Examples are given to illustrate the application of this method.

We consider the traveling wave solution $u = u(\xi)$ with $\xi = kx \pm \omega t$ and study a general nonlinear equation of the form

$$f_1(u)u_t + f_2(u)u_x + f_3(u)u_{tt} + f_4(u)u_{xx} + f_5(u)u_x^2 + f_6(u)u_xu_{xx} + f_7(u)u_{xxx} + \cdots = f_8(u) \tag{1}$$

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where $u_t = \partial u/\partial t$, $f_i(u)$, $1 \le i \le 8$, etc., are polynomials in u, and k and ω are undetermined parameters.

We make the following transformation:

$$u(x, t) = u(\xi) = AF^n \tag{2}$$

where

$$F = \frac{1}{1 + e^{\xi}} \tag{3}$$

and A and n are undetermined constants. So we get

$$u_{\xi} = nAF^{n+1} - nAF^n \tag{4}$$

$$u_{\xi\xi} = n(n + 1)AF^{n+2} - n(2n + 1)AF^{n+1} + n^2AF^n \tag{5}$$

$$u_{\xi\xi\xi} = n(n + 1)(n + 2)AF^{n+3} - n(n + 1)(3n + 3)AF^{n+2} + n(3n^2 + 3n + 1)F^{n+1} - n^3AF^n \tag{6}$$

Equation (1) becomes ($u' = u_{\xi}$)

$$\pm\omega f_1(u)u' + kf_2(u)u' + \omega^2 f_3(u)u'' + k^2 f_4(u)u'' + k^2 f_5(u)u'^2 + k^3 f_6(u)u'u'' + k^3 f_7(u)u''' + \cdots = f_8(u) \tag{7}$$

By a leading order analysis, we can determine n from equation (7). Substituting (2)–(6) into (7) and collecting terms with the same power of F, we can give the values of k, ω, and A. Thus we can obtain the correspondingly solitary wave solution.

Now we give some examples.

Example 1. Let us consider the KdV–Burgers equation

$$u_t + uu_k + \alpha u_{xx} + \beta u_{xxx} = 0 \tag{8}$$

Substituting the traveling wave solution into (8), we have

$$\pm cu' + kuu' + \alpha k^2 u'' + \beta k^3 u''' = 0 \tag{9}$$

By a leading order analysis, we set

$$u = AF^2 \tag{10}$$

Considering (10) and the corresponding u', u'' and u''', we obtain

$$\pm c(2AF^3 - 2AF^2) + kAF^2(2AF^3 - 2AF^2) + \alpha k^2(6AF^4 - 10AF^3 + 4AF^2) + \beta k^3(24AF^5 - 54AF^4 + 38AF^6 - 8AF^2) = 0 \tag{11}$$

Collecting terms with the same power of F yields

$$F^5:\quad 2kA^2 + 24\beta k^3 A = 0 \tag{12}$$

$$F^4:\quad -2kA^2 - 54\beta k^3 A + 6\alpha k^2 A = 0 \tag{13}$$

$$F^3:\quad \pm 2cA - 10\alpha k^2 A + 38\beta k^3 A = 0 \tag{14}$$

$$F^2:\quad \mp 2cA + 4\alpha k^2 A - 8\beta k^3 A = 0 \tag{15}$$

From (12)–(15), we find

$$k = \frac{\alpha}{5\beta} \tag{16}$$

$$A = -\frac{12}{25}\frac{\alpha^2}{\beta} \tag{17}$$

$$C = \mp\frac{6\alpha^2}{125\beta^2} \tag{18}$$

So we obtain the exact and explicit solitary wave solutions

$$\begin{aligned} u(x, t) &= u(\xi) = u(kx \pm \omega t) \\ &= -\frac{12}{25}\frac{\alpha^2}{\beta}\left\{1 + \exp\left[\frac{\alpha}{5\beta}\left(x \mp \frac{6\alpha^2}{25\beta}t\right)\right]\right\}^{-2} \\ &= -\frac{6}{25}\frac{\alpha^2}{\beta}\left\{1 - \tanh\left[\frac{\alpha}{10\beta}\left(x \mp \frac{6\alpha^2}{25\beta}t\right)\right]\right\}^{1/2} \end{aligned} \tag{19}$$

Example 2. Let us consider the MKdV–Burgers equation

$$u_t + u^2 u_k + \alpha u_{xx} + \beta u_{xxx} = 0 \tag{20}$$

For equation (20) we obtain the following exact and explicit solitary wave solutions:

$$\begin{aligned} u(x, t) &= u(\xi) = u(kx \pm \omega t) \\ &= \pm\left(-\frac{2}{3}\frac{\alpha^2}{\beta}\right)^{1/2}\left\{1 + \exp\left[\frac{\alpha}{3\beta}\left(x \mp \frac{2\alpha^2}{9\beta}t\right)\right]\right\}^{-1} \\ &= \pm\left(-\frac{\alpha^2}{6\beta}\right)^{1/2}\left\{1 - \tanh\left[\frac{\alpha}{6\beta}\left(x \mp \frac{2\alpha^2}{9\beta}t\right)\right]\right\} \end{aligned} \tag{21}$$

Example 3. Let us consider the combined KdV–MKdV equation

$$u_t + uu_k + u^2 u_k + \beta u_{xxx} = 0 \tag{22}$$

For equation (22) we obtain the following exact and explicit solitary wave solutions:

$$u(x, t) = u(\xi) = u(kx \pm \omega t)$$

$$= \pm\left\{1 + \exp\left[\pm\left(-\frac{1}{6\beta}\right)^{1/2}\left(x - \frac{1}{6}t\right)\right]\right\}^{-1}$$

$$= \pm\frac{1}{2}\left\{1 - \tanh\left[\pm\left(-\frac{1}{24\beta}\right)^{1/2}\left(x - \frac{1}{6}t\right)\right]\right\} \tag{23}$$

Example 4. Let us consider the Newell–Whitehead equation (Cariello and Tabor, 1989)

$$u_t = u_{xx} + u - 2u \tag{24}$$

For equation (24) we obtain the following exact and explicit solitary wave solutions:

$$u(x, t) = u(\xi) = u(kx \pm \omega t)$$

$$= \pm\frac{1}{\sqrt{2}}\left\{1 + \exp\left[\pm\frac{1}{2\sqrt{2}}\left(x - \frac{3}{\sqrt{2}}t\right)\right]\right\}^{-1}$$

$$= \pm\frac{1}{2\sqrt{2}}\left\{1 - \tanh\left[\pm\frac{1}{2\sqrt{2}}\left(x - \frac{3}{\sqrt{2}}t\right)\right]\right\} \tag{25}$$

Example 5. Let us consider the dissipative Φ^4-model equation (Lee, 1988)

$$u_{xx} - \gamma u_t - u_{tt} = \alpha u + \beta u^3 \tag{26}$$

For equation (26) we obtain the following exact and explicit solitary wave solutions:

$$u(x, t) = u(\xi) = u(kx \pm \omega t)$$

$$= \pm\left(-\frac{\alpha}{\beta}\right)^{1/2}\left\{1 + \exp\left[\pm\frac{1}{2}\frac{\alpha^{1/2}(9\alpha - 2\gamma)^{1/2}}{\gamma^{1/2}}x \pm \frac{3\alpha}{2\gamma}t\right]\right\}$$

$$\pm\frac{1}{2}\left(-\frac{\alpha}{\beta}\right)\left\{1 + \tanh\left[\pm\frac{\alpha^{1/2}(9\alpha - 2\gamma)^{1/2}}{4\gamma^{1/2}}x \pm \frac{3\alpha}{4\gamma}t\right]\right\} \tag{27}$$

Example 6. Let us consider the generalized Fisher equation (Yang *et al.*, 1994)

$$u_t - \alpha u_{xx} = \beta u - \gamma u^2 - \delta u^3 \tag{28}$$

For equation (28) we obtain the following exact and explicit solitary wave solution:

$$
\begin{aligned}
u(x, t) &= u(\xi) = u(kx \pm \omega t) \\
&= -\frac{\gamma \mp (4\beta\delta + \gamma^2)^{1/2}}{2\delta}\left\{1 + \exp\left[\mp\frac{\gamma \mp (4\beta\delta + \gamma^2)^{1/2}}{(8\alpha\delta)^{1/2}}x \right.\right. \\
&\quad \left.\left. \pm\left(\beta + \frac{\gamma^2 + 2\beta\delta \mp \gamma(4\beta\delta + (8\alpha\delta)^{1/2}}{4\delta}t\right)\right]\right\}^{-1} \\
&= -\frac{\gamma \mp (4\beta\delta + \gamma^2)^{1/2}}{4\delta}\left\{1 \mp \tanh\left[\frac{\gamma \mp (4\beta\delta + \gamma^2)^{1/2}}{(32\alpha\delta)^{1/2}}x \right.\right. \\
&\quad \left.\left. \pm\left(\frac{\beta}{2} + \frac{\gamma^2 + 2\beta\delta \pm \gamma(4\beta\delta + \gamma^2)^{1/2}}{8\delta}t\right)\right]\right\}
\end{aligned}
\tag{29}
$$

Example 7. Let us consider the elastic-medium wave equation (Drazin and Johnson, 1989):

$$u_{tt} - u_{xx} - u_x u_{xx} - u_{xxx} = 0 \tag{30}$$

For equation (30) we obtain the following exact and explicit solitary wave solution:

$$
\begin{aligned}
u(x, t) &= u(\xi) = u(kx \pm \omega t) \\
&= 6(\sqrt{2} \mp (2\sqrt{2 + 8c^2})^{1/2}\left\{1 + \exp\left[\pm\frac{-1 \pm \sqrt{1 + 4c^2}}{\sqrt{2}}x \pm ct\right]\right\} \\
&= 3(\sqrt{2} \pm \sqrt{2 + 8c^2})^{1/2}\left\{1 \pm \tanh\left[\frac{1 \pm \sqrt{1 + c^2}}{2\sqrt{2}}x - ct\right]\right\}
\end{aligned}
\tag{31}
$$

Example 8. Let us consider the Fitzbugh–Nagumo equation (Nacci and Clarkson, 1992)

$$u_t - u_{xx} - u(1 - u)(u - \alpha) = 0 \tag{32}$$

For equation (32) we obtain the following exact and explicit solitary wave solution:

$$
\begin{aligned}
u(x, t) &= u(\xi) = u(kx \pm \omega t) \\
&= \pm 2\left(\frac{\alpha}{1 - 2\alpha^2}\right)^{1/2}\left\{1 + \exp\left[\pm\left(\frac{\alpha}{1 - 2\alpha^2}\right)^{1/2}x - \alpha\left(\frac{1 + 2\alpha^2}{1 - 2\alpha^2}\right)t\right]\right\}^{-1} \\
&= \pm\left(\frac{\alpha}{1 - 2\alpha^2}\right)^{1/2}\left\{1 + \tanh\left[\pm\left(\frac{\alpha}{2(1 - 2\alpha^2}\right)^{1/2}x - \left(\frac{\alpha(1 + 2\alpha^2)}{2(1 - 2\alpha^2)}\right)t\right]\right\}
\end{aligned}
\tag{33}
$$

In summary, exact and explicit solitary wave solutions for nonlinear evolutions and wave equations have been obtained. These solutions were first obtained or given by using other methods. The method presented here is not only general and effective, but also concise and primary. We can easily extend the method to multidimensional nonlinear equations and nonlinear equation systems, which will be published elsewhere.

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